Globally linked vortex clusters in trapped wave fields

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We put forward the existence of a rich variety of fully stationary vortex structures, termed *H* clusters, made of an increasing number of vortices nested in paraxial wave fields confined by trapping potentials. However, we show that the constituent vortices are *globally linked*, rather than products of independent vortices. Also, they always feature a *monopolar* global wave front and exist in nonlinear systems, such as the Bose-Einstein condensates. Clusters with multipolar global wave fronts are nonstationary or, at best, flipping.

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Singular wave structures, which contain topological wave front dislocations [1], are ubiquitous in many branches of classical and quantum science. Screw dislocations, or vortices, are a common dislocation type. Wave packets with nested vortices find applications in fields as diverse as cosmology, biosciences or solid state physics [2–5]. As striking examples, they are at the heart of schemes to generate engineered quNits in quantum information systems in higher-dimensional Hilbert spaces [6,7], are believed to be essential for the onset of superfluidity in Bose-Einstein condensates (BECs) [8–11], or allow tracking the motion of a single atom [12].

A recent study of the motion of vortex lines governed by both linear and nonlinear Scrödinger equations describing the dynamics of atoms in harmonic traps revealed that the topological features of vortex dynamics are, to a large extent, universal [13]. The dynamics of the vortices nested on localized wave packets depends on the evolution of the host beam, and on the interferences and interactions between the vortices [14]. Multiple vortices nested on the same host typically follow dynamical evolutions that might include large vortex drifts that destroy their initial arrangement, and vortex-pair annihilations that destroy the vortices themselves. Vortex evolutions are particularly complex in strongly nonlinear media, such as the BECs, where the vortices can interact with each other. Therefore, a fundamental question arises about whether stationary or quasistationary vortex clusters or lattices [15] made of vortices with equal and with opposite topological charges exist. To isolate the pure vortex features from the dynamics solely induced by the evolution of the host wave packet, it is convenient to study wave fields confined by suitable potentials, as in weakly interacting trapped BECs.

In this paper we show that vortex clusters with multipolar global wave fronts nested in wave fields confined by trapping potentials are nonstationary, when the number of vortices and their location are not constant during dynamical evolution, or at best flipping, when the vortices periodically flip their topological charges through extremely sharp Berry trajectories [16]. In the former case, multiple vortex revivals mediated by Freund stationary point bundles [17], which carry the necessary Poincaré-Hopf indices [18], can occur. In contrast, we find that a rich variety of *fully stationary* vortex clusters made of an increasing number of vortices do exist. The important point we put forward is that these clusters are *globally linked*, rather than products of independent vortices. Also, they feature a *monopolar* global wave front. We also show that the clusters exist, and are robust in nonlinear systems such as the interacting BECs.

We thus address the slowly varying evolution of generic wave functions governed by the paraxial wave equation

$$iA_z = \mathcal{L}A + \mathcal{N}(A), \tag{1}$$

where A is a complex field, $\mathcal{L}$ is a two-dimensional linear differential operator containing a trapping potential, and $\mathcal{N}(A)$ takes care of any nonlinear contribution. We assume the trapping potential to be harmonic, thus $\mathcal{L} = -\frac{1}{2}(\partial_x^2 + \partial_y^2) + (n_x x^2 + n_y y^2)$. To be specific, when $\mathcal{N}(A) \sim |A|^2 A$, this equation models the propagation of a light beam guided in a Kerr nonlinear graded-index medium and the mean-field evolution of a two-dimensional trapped BEC at zero temperature (where $n_{x,y}$ are proportional to the trap frequencies in appropriate units). Here we will consider only the symmetric case; hence $n_x = n_y = 2$. For convenience, from now on we will split $A(x,y;z) = F(x,y;z)V(x,y;z)$, taking the host packet F as given by the fundamental mode of the trapping potential, $F(x,y;z) = \exp(-x^2 - y^2)\exp(-2iz)$. In the linear case, the function $V(x,y;z)$ carries all the essential information about the solutions and, in particular, about vortex dynamics. Here we will consider the evolution of polynomial initial data for V corresponding to (multi)vortex so-

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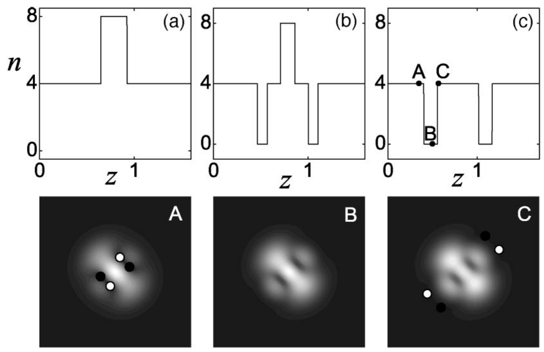

FIG. 1. Evolution of a vortex quadrupole, constructed as the product of four single-charge vortices. Upper row shows number of vortices (n) as a function of z for different values of the initial quadrupole size. (a) $a=0.5$, (b) $a=1.1$, and (c) $a=1.2$. Bottom row shows intensity snapshots corresponding to points labeled A, B, and C in (c). Black-filled circles represent positive vortices; white-filled circles represent negative vortices.

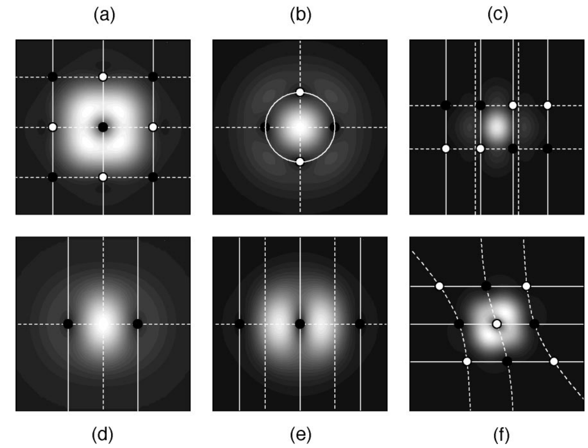

FIG. 2. H-cluster "zoology." Shown are several examples of stationary vortex matrices and vortex arrays that can be constructed (see text for details). Lines show zero crossings of Re(V) (full lines) and Im(V) (dashed). Features are as in Fig. 1.

lutions of Eq. (1). Such solutions can be expressed as finite series and, as will be clear later, all of them must be periodic or stationary.

Let us first consider the linear $[\mathcal{N}(A)=0]$ evolution of vortex clusters built as products of n independent single-charge vortices, $V(x,y;z=0)=\Pi_{k=1}^{n}[x-x_k+i\sigma_k(y-y_k)]$, where (x_k,y_k) are the locations in the vortex cores in the transverse plane, and $\sigma_k=\pm 1$. None of the above product vortex clusters is found to be dynamically stationary. On the contrary, the number of vortices and their location is found to vary during evolution, so that the initial vortex structure is destroyed. These results can be illustrated by examining the evolution of the four-vortex cluster, $V(z=0)=(x+a+iy)\times(x-a+iy)(x-iy-ia)(x-iy+ia)$, which contains two vortices with positive topological charge and two vortices with negative charge in a symmetrical geometry. This cluster features a quadrupolar global wave front, as is revealed by calculating the gradient of the wave front Φ far from the cluster core, to obtain $|\nabla\Phi|\sim 1/\rho^3$, where ρ is the polar coordinate, similar to the corresponding electrostatic multipole [19]. Substitution into Eq. (1) yields

$$V(x,y;z)=[(x^2+y^2)(x^2+y^2+2e^{4iz}-2)]e^{-8iz}+4ia^2xye^{-4iz}+\frac{1}{2}(1-e^{-4iz})^2-a^4. \quad (2)$$

One finds three different regimes of evolution, as shown in Fig. 1: vortex drifts, vortex-pair annihilations, and revivals take place, so that depending on the value of the geometrical parameter a, the total number of vortices, n, hosted in the wave field during propagation can oscillate between (i) 4 and 8 [see Fig. 1(a)]; (ii) 4, 0, and 8 [see Fig. 1(b)]; and (iii) 4 and 0 [see Fig. 1(c)]. When $n=0$, it is understood that all vortices have annihilated each other. Analogous evolutions were found for octupoles and higher-order multipoles. Only dipoles can be made quasistationary, but are made flipping when the corresponding vortex twins periodically flip their topological charges. Thus, the main conclusion reached is that the interference between the constituent vortices of all the product clusters produces beatings between the normal modes of the system, rendering the clusters nonstationary.

The key insight we put forward in this paper is that such beatings are not associated to the intrinsic or local properties of the individual vortices, but to the very way the vortices are *globally linked* in the host wave packet. As an example, let us consider the evolution of

$$V(x,y;z=0)=x^2+y^2-a^2+2ixy, \quad (3)$$

which contains four vortices located at the same positions and having the same charges as those of the vortex quadrupole considered above [see Fig. 2]. However, in this cluster the vortices are intimately linked to each other, rather than individually nested in the host F. This fact manifests itself in the global wave front of the cluster, which behaves as $|\nabla\Phi|\sim\cos(2\phi)/\rho+O(1/\rho^3)$, and thus features almost everywhere a *monopolar* decay. In this case, the vortex evolution is given by

$$V(x,y;z)=(x^2+y^2-1/2+2ixy)e^{-4iz}+1/2-a^2, \quad (4)$$

an evolution when the cluster is constructed with $a=1/\sqrt{2}$ does become fully stationary.

The above four-vortex cluster is not an isolated case, but rather an example of whole existing families of fully stationary vortex structures made of globally linked vortices. In fact, the solutions of Eq. (1) with $\mathcal{N}(A)=0$ have the form

$$A(x,y;z)=\sum_{k,l=0}^{\infty} C_{kl}H_k(x\sqrt{2})H_l(y\sqrt{2})e^{-x^2-y^2}e^{-2i(k+l+1)z}, \quad (5)$$

where H_j are the Hermite polynomials. Therefore, the evolution of initial data of the form $V(x,y;z=0)=\Sigma_{k=0}^{n}C_kH_k(\xi)H_{n-k}(\eta)$ for any $C_k\in\mathbf{C}$, where $\xi=x\sqrt{2}$ and $\eta=y\sqrt{2}$, is given by

$$V(x,y;z)=e^{-2inz}\sum_{k=0}^{n} C_k H_k(\xi)H_{n-k}(\eta). \tag{6}$$

On physical grounds, this simple mathematical result shows that all the stationary clusters are made of globally linked vortices. Equation (6) allows us to build a variety of structures, to be termed Hermite, or *H* clusters, whose key features are discussed in what follows.

Perhaps the simplest types of *H* clusters are those with a $n\times n$ *matrix* geometry, thus containing n^2 vortices. These clusters can be generated by using as initial data, for example, $V_{n\times n}(x,y;z=0)=H_n(\xi)+iH_n(\eta)$. In this function the vortex charges alternate throughout the matrix and the vortex locations are dictated by the zeros of the particular Hermite polynomials involved. In general, these vortex matrices are not regular, the distance between vortices varies along the matrix. However, in the particular cases with $n=2$ and $n=3$, the matrix is regular. The $n=3$ case is shown in Fig. 2(a). Notice that a 2×2 matrix cluster can be generated either with $V(x,y;z=0)=H_2(\xi)+iH_2(\eta)$ or with $V(x,y;z=0)=H_2(\xi)+H_2(\eta)+iH_1(\xi)H_1(\eta)$. Actually, this latter possibility generates the stationary four-vortex cluster discussed earlier [see Fig. 2(b)].

One can also build $m\times n(m\neq n)$ stationary vortex matrices. A possible choice for the vortex function generating such a vortex matrix is $V_{m\times n}(x,y;z=0)=H_m(\xi)+iH_{|m-n|}(\xi)H_n(\eta)$. We show the 4×2 vortex matrix as an example in Fig. 2(c). In contrast to the $n\times n$ matrices, in the general case the topological charges carried by the vortices of the $m\times n$ matrices do not alternate sign throughout the matrix. An important subclass of the $m\times n$ vortex matrices are the $m\times 1$ cases, to be termed as *vortex arrays*. They consist in *m* collinearly displaced vortices of the same topological charge. Figures 2(d) and 2(e) show illustrative examples. The simplest array is the vortex twin shown in Fig. 2(d). A pair of identical vortices that, contrary to the vortex dipole that either undergoes periodic annihilations and revivals or charge flip-flops, can be made fully stationary.

The $n\times n$ matrices are either chargeless for even *n*, or carry a single net charge for odd values of *n*, while the $m\times 1$ arrays carry a total topological charge of *m*. In any case, the wave front of all the *H* clusters is found *to feature a monopolar* decay ($\sim 1/\rho$) almost everywhere.

More complex *H* clusters also exist, and a full classification of all the possibilities falls beyond the scope of this paper. However, an example of one of such exotic *H* clusters is displayed in Fig. 2(f) that corresponds to the cluster built with $V(x,y;z=0)=H_3(\eta)+i[H_3(\xi)+H_1(\eta)H_2(\xi)]$. A rich variety of possibilities contained in Eq. (6) is clearly apparent.

An interesting issue is the existence and stability of vortex clusters in the presence of nonlinear cubic interactions, such as those appearing in the propagation of beams in Kerr media or in the dynamics of BECs. To ease the comparison with BEC literature, we choose now $n_x=n_y=1/2$ and $\mathcal{N}(A)=U|A|^2A$ (Ref. [20]). In this context, the evolution variable is denoted by *t* instead of *z*. With this choice of parameters, the range of *U* values experimentally accessible for the two-

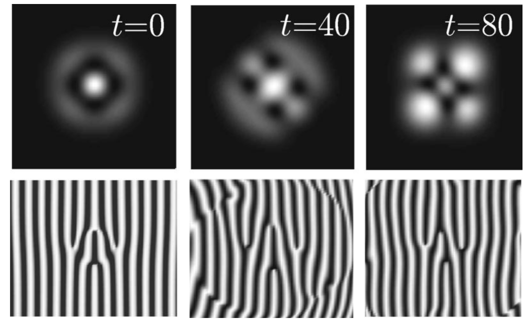

FIG. 3. Stable evolution of initial data given by Eq. (3) for $UN=10$. Upper row shows intensity plots; bottom row shows interference fringes. Spatial region spanned is $[-4,4]\times[-4,4]$.

dimensional case is $0<U<10^2$–10^3 (Ref. [21]).

We have studied several particular examples to verify that these structures indeed exist and are stable in the nonlinear regime. We have taken as initial data several linear configurations, such as a single vortex, dipole systems, and the four-vortex cluster given by Eq. (3), and evolved them for $UN=10$ ($N=\int|A|^2 d\mathbf{x}$ is the wave function norm) using a standard split-step integrator. It is found that although the background performs oscillations and the vortex locations oscillate around their equilibrium positions, the vortex clusters remain stable (see Fig. 3). We have also searched for stationary solutions of Eq. (1), of the form $A(x,y;t)=e^{i\lambda t}\psi(x,y)$. To do so we have used a steepest descent method to minimize the functional [22]

$$F(\psi)=\frac{\int \psi^*(-\Delta-\lambda+r^2+U|\psi|^2)\psi d\mathbf{x}}{\int |\psi|^2 d\mathbf{x}}, \tag{7}$$

whose minima (except for $\psi=0$) coincide with the stationary solutions of Eq. (1) for a given value of λ. For instance,

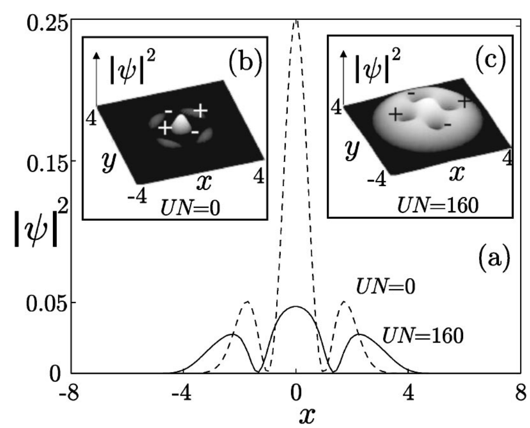

FIG. 4. Linear four-vortex cluster [Eq. (3)] vs its nonlinear stationary version for $U=100$, $\lambda=8$. (a) Plots of $|\psi(x,y=0)|^2$ for the linear (dashed line) and nonlinear (solid line) cases. (b,c) Surface plots of $|\psi(x,y)|^2$ for (b) the linear and (c) nonlinear situations. The vortex locations and topological charges are indicated by plus and minus signs.

taking the linear four-vortex cluster as initial data for the minimization process and setting $\lambda = 8.0$ and $U = 100$, we found a stationary four-vortex cluster solution (see Fig. 4) with norm $N = \int |\psi|^2 d\mathbf{x} \simeq 1.6005$ (thus the product $UN \simeq 160$ which lies into the fully nonlinear regime). We have verified that this solution is robust under time evolution when small perturbations are added. These evidences show that the existence of H clusters in a BEC should be experimentally accessible, at least from the dynamical point of view.

To conclude, we stress that the constituent vortices of the H cluster are *globally linked*, rather than products of independent vortices. Following this idea, it is possible to generate a variety of additional novel structures with fascinating properties. A good example is the circular vortex necklace generated at the intersection between the circle $x^2 + y^2 - a^2 = 0$, where $\mathrm{Re}(V) = 0$, and the lines $y \pm \tan(2k\pi/n)x = 0$, $k \in \mathbf{N}$, where $\mathrm{Im}(V) = 0$. Those are quasistationary, purely flipping clusters made of n vortices (see Fig. 5), a feature so far only known to occur with vortex dipoles. Once again, the vortices forming the necklace are intimately linked and do not exhibit a n-polar wave front. The exploitation of such intrinsic linking might open new opportunities in classical and quantum systems based on topological light and matter waves. The major challenge is the demonstration of the generation of the clusters by suitable computer-generated holograms [23] in optics and phase-imprinting techniques in BECs [24].

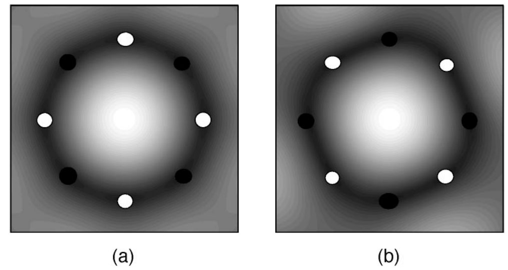

FIG. 5. Evolution of a flipping $n = 8$ circular vortex necklace. Vortex patterns at (a) $z = 0$ and (b) $z = 3\pi/16$.

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